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Journal of Geometry and Physics 53 (2005) 421–427

JOURNAL OF GEOMETRY AND PHYSICS

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Lobachevsky triangle altitudes theorem as the Jacobi identity in the Lie algebra of quadratic forms on symplectic plane

V. Arnold*

Ceremade, Université Paris Dauphine, Paris, France, V. A. Steklov Institute of Mathematics, Moscow, Russia

Received 7 July 2004; accepted 22 July 2004
Available online 11 September 2004

Abstract

An isomorphism between the Lobachevsky and de Sitter's world geometries with the symplectic geometry and the Lie algebra of binary quadratic forms is used to derive the altitudes concurrence for the Lobachevsky and de Sitter triangles.

Keywords: Projective geometry; de Sitter world; Symplectic algebra; Klein model

The Lobachevsky plane may be considered [1] as the projectivized version of the space of the positive definite binary quadratic forms. This disc in $\mathbb{R}P^2$ forms the Klein model of the Lobachevsky plane, the complementary Möbius band forming the de Sitter world of the hyperbolic binary forms on the same plane (considered also up to a multiplication by a nonzero constant). The metrics are defined as the determinant's second differential form on the determinant one (minus one) forms hyperboloids.

This Lobachevsky plane Riemannian metrics and this de Sitter's world Lorentzian pseudoriemannian metrics are invariant under the group $\mathrm{SL}(2,\mathbb{R})$ of the symplectic

* Corresponding author.
E-mail address: arnold@ceremade.dauphine.fr (V. Arnold).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2004.07.008

linear mappings of the plane $\{(p, q)\}$. The quadratic forms on this symplectic plane, $ap^2 + 2bpq + cq^2$, are forming the 3-space $\mathbb{R}^3$ with coordinates a, b, c. The group action preserves the (quadratic) determinant of the form, $\Delta = ac - b^2$, and the Lobachevsky plane is the projectivization of the conic space $\Delta > 0$, while the de Sitter world is the projectivization of the complementary conic space, $\Delta < 0$. The cone $\Delta = 0$ projective version is the absolute circle, bounding the Klein model disc.

We shall therefore consider the points of the Lobachevsky plane and of the de Sitter world as "forms" (a "form" $[a : b : c]$ being the quadratic form $ap^2 + 2bpq + cq^2$, considered up to a nonzero constant factor).

The points of the de Sitter world can also be interpreted as the straight lines of the Lobachevsky plane (and viceversa): the two tangent lines from this de Sitter point of $\mathbb{R}P^2$ to the absolute circle define two tangency points on the absolute, and the Lobachevsky line, identified with the de Sitter point, joins them (being its projectively dual line for the duality defined by the absolute).

Similarly, the Lobachevsky lines, containing a common Lobachevsky disc point, can be interpreted as the de Sitter world points, at each of which intersect the two tangent lines of the absolute circle at its two points on a Lobachevsky line, chosen among those lines, containing the original Lobachevsky point.

All the Lobachevsky lines, passing through a given point of the Lobachevsky disc, are therefore interpreted as forming a curve in the de Sitter world. This curve is the de Sitter world straight line (which is a projective line, not intersecting the Lobachevsky disc, in $\mathbb{R}P^2$). This line is projectively dual to the original point of the Lobachevsky disc (in the sense of the projective duality, defined by the absolute circle).

The goal of the present article is to translate the altitudes intersection property of the Lobachevsky triangles to this quadratic forms geometry[1].

The space of quadratic forms on the symplectic plane $\mathbb{R}^2$ (with Darboux coordinates p, q of the symplectic structure $\omega = dp \wedge dq$) is the Poisson brackets Lie algebra (since the Poisson bracket of two quadratic forms is a quadratic form). The group $\mathrm{SL}\,(2, \mathbb{R})$ acts linearly on the space $\mathbb{R}^3$ of the quadratic forms on $\mathbb{R}^2$.

We start with the interpretation of the Poisson brackets in the terms of the geometry of the Lobachevsky plane of positive definite quadratic "forms" and of the de Sitter plane of the hyperbolic "forms". This Poisson bracket operation $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is intrinsic (invariant under the above group action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^3$).

Theorem 1. *The Poisson bracket form of two positive definite forms is the hyperbolic form, represented by the straight line, joining the original two points of the Lobachevsky plane in Lobachevsky geometry.*

This bracket form is, therefore, hyperbolic.

Proof. We shall use the explicit expression (1) of the scalar product of quadratic forms, polar to the absolute circle equation $\Delta = 0$, where $\Delta(\xi = ap^2 + 2bpq + cq^2) = ac - b^2$.

[1] The Euclidean triangle altitudes concurrence property interpretation in terms of the SO(3) Lie algebra (of vector products in the oriented Euclidean 3-space) Jacobi identity had been explained by the author many years ago; this Euclidean property proof has contained already the present paper Theorem 4.

The polar scalar product is a symmetric bilinear form on the space of the binary quadratic forms, coinciding with Δ on the diagonal:

$$2\tilde{\Delta}(\xi', \xi'') = a'c'' + c'a'' - 2b'b''. \tag{1}$$

For two elliptic quadratic forms, $A = \alpha p^2 + \beta q^2$, $B = \gamma p^2 + \delta q^2$, we find their Poisson bracket to be the hyperbolic form

$$\{A, B\} = 4(\alpha\delta - \beta\gamma)pq. \tag{2}$$

The corresponding projective line connects the points A and B, since formulas (1) and (2) show that

$$\tilde{\Delta}(A, \{A, B\}) = \tilde{\Delta}(B, \{A, B\}) = 0. \tag{3}$$

This example of the pair (A, B) suffices for the proof of Theorem 1, since any two elliptic forms are diagonalizable by the same choice of symplectic coordinates. □

Theorem 2. *The Poisson bracket form of a positive definite form with a hyperbolic form is represented by the straight line of the Klein model, which is orthogonal to the Lobachevsky straight line, corresponding to the hyperbolic original form, and goes through the Lobachevsky plane point, corresponding to the original positive definite form.*

Proof. We use once more the identity (2), choosing $\alpha\beta > 0$ and $\gamma\delta < 0$ to represent the positive definite form A and the hyperbolic form B. The eigenvectors theory shows that this example is universal, since the positive definiteness of A suffices for the common diagonalization of A and B by the same choice of symplectic coordinates.

The two equations (3) show, in this case, that the line corresponding to $\{A, B\}$ contains the point A (the first equation) and is orthogonal to the line B (the second equation). To deduce the orthogonality from the second Eq. (3) is easy, but, anyway, the explicit form of the Poisson bracket (2) proves this orthogonality. The projective meaning of this orthogonality condition for two Lobachevsky lines in the Klein model is that each of the lines passes through the point, dual to the other line (with respect to the duality defined by the absolute circle). □

Theorem 3. *The Poisson bracket form of two hyperbolic forms corresponds to the intersection point of the two projective lines corresponding to the two original hyperbolic forms.*

Proof. The two hyperbolic forms eigenvalues theory shows[2] that they are either diagonalizable by the same symplectic basis choice, or reducible to the special pair of forms,

$$A = p^2 - q^2, \quad B = \lambda pq.$$

In the first case the bracket is hyperbolic and the corresponding projective plane point belongs to both dual straight lines, according to the identity (3).

[2] It can be deduced from the Lobachevsky geometry as well: it means that a pair of intersecting Lobachevsky lines is defined, up to a Lobachevsky motion, by their angle.

In the second case the Poisson bracket calculation provides the answer

$$\{A, B\} = 2\lambda(p^2 + q^2), \tag{4}$$

which form is elliptic, corresponding, therefore, to an intersection point in the Lobachevsky disc of the Klein model. This point belonging to the lines, dual to the forms A and B, is proved by the identities (1) and (4):

$$2\tilde{\Delta}(A, \{A, B\}) = 2\lambda - 2\lambda = 0,$$
$$2\tilde{\Delta}(B, \{A, B\}) = -2\tfrac{1}{2}\lambda \cdot 0 = 0. \qquad \square$$

Theorem 4. *If three nonzero vectors of* $\mathbb{R}^3$ *verify the identity* $f + g + h = 0$, *the three corresponding points of the projective plane lie on a projective straight line, and the corresponding three projective lines (of the dual projective plane) have a common point.*

Proof. The relation $f + g + h = 0$ for some representative vectors of three points of the projective plane means the existence of a linear dependence between *any* three representatives of the points, proving the first statement of the theorem. The second statement follows from it, since the duals of three points, belonging to a line, are three lines, containing the point, dual to this line. $\square$

Problem. Find the geometrical condition of two forms, one elliptic an the other hyperbolic, equivalent to the belonging of the point, corresponding to the elliptic one, to the line, corresponding to the hyperbolic one.

Answer: The zero directions of the hyperbolic form should be orthogonal in the metrics defined by the elliptic form.

Theorem 5. *The three altitudes of a Lobachevsky plane triangle are concurrent (have a common intersection point).*

Proof. Denote A, B and C the side *lines* of the triangle. We denote the same way the corresponding three (hyperbolic) quadratic forms. Their Poisson brackets

$$\{A, B\} = c, \quad \{B, C\} = a, \quad \{C, A\} = b$$

are the (elliptic) forms, representing the three vertices of the original triangle (according to Theorem 3).

The standard notation of this triangle would be (a, b, c), the vertex a being opposite to the side A (and so on).

Consider the altitudes of this triangle. The line, passing by a and orthogonal to the side $bc = A$, is, according to Theorem 2, the "form", which corresponds to the Poisson bracket of a and of A,

$$(\text{altitude from } a \text{ to } A) \sim (\{\{B, C\}, A\}).$$

The three altitudes of the triangle (a, b, c) *are thus the geometrical versions of the three quadratic forms*

$$(f, g, h) = (\{\{B, C\}, A\}, \quad \{\{C, A\}, B\}, \quad \{\{A, B\}, C\}).$$

According to the Jacobi identity (for the Poisson brackets Lie algebra of the quadratic forms on the symplectic plane with Darboux coordinates (p, q)), the sum of these three quadratic forms is the zero form, $f + g + h = 0$.

Applying Theorem 4 to these three forms, we conclude that the three altitudes lines have a common point in $\mathbb{R}P^2$, proving Theorem 5. □

In the case, where the triangle (abc) angles are less than $\pi/2$, the intersection points lie inside the triangle, and hence do belong to the Lobachevsky plane part of the projective plane[3].

It is easy to provide an example of a triangle (with an angle greater than $2\pi/3$), for which no altitudes have a common point in the Lobachevsky disc.

In this case our Theorem 5 shows the existence of a common intersection point of all the three altitudes projective lines in the de Sitter world (or on the separating absolute).

Remark. Our Theorem 5 proves also the altitudes concurrence for the *de Sitter triangles and for the mixed triangles*, some of whose vertices (and or sides) are Lobachevsky plane points and lines, the others being de Sitterian (belonging in the separating case to the absolute for the vertices, or being tangent to it for the sides).

To avoid the angles in the altitudes description, we might call two projective lines of the Klein model orthogonal, if one (each) of them contains the point, dual to the other one. In the Lobachevsky disc, it is just the Lobachevsky orthogonality definition (providing also the de Sitter Lorentzian metrics orthogonality, when the intersection point is in the de Sitter part of the Klein model).

Reformulating the altitudes concurrence theorem in the de Sitter or mixed triangles case, we might eliminate the Lobachevsky geometry mentioning, formulating the results as several elementary projective geometry statements. One might use the Jacobi identity and other theorems of the quadratic forms symplectic algebra to obtain new results of projective geometry.

Example. Consider three nonintersecting Lobachevsky lines $((AB), (CD), (EF))$, represented in the Klein model by three chords of the Lobachevsky disc, whose endpoints follow the absolute circle in the order $(ABCDEF)$ (see Fig. 1).

Construct the three intersection points

$$(CD) \cap (EF) = \alpha, \quad (EF) \cap (AB) = \beta, \quad (AB) \cap (CD) = \gamma$$

[3] One might replace $\pi/2$ by $2\pi/3$, (if all the angles are smaller than $2\pi/3$, the altitudes of a Lobachevsky triangle intersect inside the Lobachevsky disc, while for any angle greater than $2\pi/3$, there exists a Lobachevsky triangle with such an angle, whose altitudes have no common Lobachevsky points). Namely, the triangles, separating the two cases (having the orthocentre on the absolute) are isometric to the triangles with vertices $\{x, -y, z = ixy/(1 + xy)\}$, $0 \le x \le 1$, $\quad 0 \le y \le 1$, therefore $|z| \le 1/2$, in the Klein model.

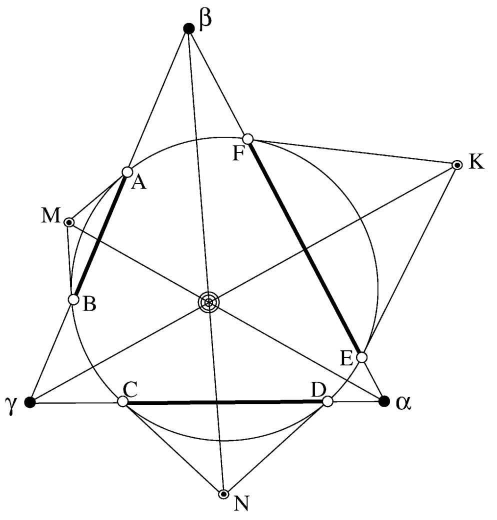

Fig. 1. The Inscribed Hexagon Theorem as the de Sitter altitudes concurrence property.

of the projective lines (these points lie inside the de Sitter part of the projective plane of the Klein model).

Consider also the three dual points

$$M = (AB)^{\vee}, \quad N = (CD)^{\vee}, \quad K = (EF)^{\vee}$$

(the point M is the intersection point of the two tangents of the absolute circle at its intersection points A and B with the chord (AB) and similarly for N and K).

Inscribed Hexagon Theorem. *The three lines* (αM), (βN) *and* (γK) *are concurrent (have a common point).*

Proof. The projective line (αM) is the altitude of the de Sitter triangle (α, β, γ), starting at point α, since the point M is dual to the side $(AB) = (\beta\gamma)$. Similarly, (BN) is the altitude, issued from β, and (γK) from γ, hence their concurrence is the altitudes concurrence property for the de Sitter triangle $(\alpha\beta\gamma)$. □

Corollary 6. *The three dual points,* $(\alpha M)^{\vee}$, $(\beta N)^{\vee}$ *and* $(\gamma K)^{\vee}$ *lie on a projective line in* $\mathbb{R}P^2$.

These three points are the intersection points of the three pairs of lines: $(KN) \cap (AB)$, $(MK) \cap (CD)$ and $(NM) \cap (EF)$.

We see that the Inscribed Hexagon Theorem is one more geometrical manifestation of the mathematical physics symplectic Jacobi identity.

Acknowledgements

This work was partially supported by RBRF, grant 02-01-00655.

Reference

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